Large-*N* theory of strongly commensurate dirty bosons: absence of a transition in two dimensions

# Large-*N* theory of strongly commensurate dirty bosons: absence of a transition in two dimensions

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**Abstract**
The spherical limit of strongly commensurate dirty bosons is studied perturbatively at weak disorder and numerically at strong disorder in two dimensions (2D). We argue that disorder is not perfectly screened by interactions and consequently that the ground state in the effective Anderson localization problem always remains localized. As a result there is only a gapped Mott insulator phase in the theory. Comparisons with other studies and the parallel with disordered fermions in 2D are discussed. We conjecture that while for the physical cases $N = 2$ (XY) and $N = 1$ (Ising) the theory should have the ordered phase, it may not for $N = 3$ (Heisenberg).

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## 1. Introduction

The problem of interacting bosons in a random potential is a paradigmatic case of an interacting disordered system and as such has attracted much attention throughout the years [1, 2]. Although in one version or another it has been used to describe numerous physical situations [3], it has proven very difficult for theoretical analysis, since it inextricably combines the effects of interactions and Anderson localization. Just like its fermionic cousin the metal–insulator transition [4], the problem of dirty bosons seems to lack a simple analytic mean-field theory around which to begin a systematic study. Most of the information on the dirty boson quantum phase transitions therefore derives from numerical studies [5] and more recently from an expansion around the lower critical dimension [6].

In this paper we will be concerned with a limited class of the dirty boson models at a commensurate filling and study the limit where the number of bosonic species *N* is large [7]. As is well known in this limit the mean-field theory, or the saddle-point approximation, becomes the exact solution. The quantum-mechanical action at $T = 0$ that defines our problem is

$$S[\Psi] = \int d^D \boldsymbol{x} \, d\tau \left\{ (\partial_\tau \Psi(\boldsymbol{x}, \tau))^2 + (\nabla \Psi(\boldsymbol{x}, \tau))^2 \right.$$
$$\left. + (V(\boldsymbol{x}) - \mu) \, \Psi^2(\boldsymbol{x}, \tau) + \frac{\lambda}{N} \Psi^4(\boldsymbol{x}, \tau) \right\} \tag{1}$$

where $\Psi(\boldsymbol{x}, \tau)$ is a real $N$-component bosonic field, $\Psi^2 = \sum_{\alpha=1}^{N} \Psi_\alpha^2$, and $V(\boldsymbol{x})$ is a random (in space) external potential. For simplicity, it will be assumed that $V(\boldsymbol{x})$ is uncorrelated, so that $\langle V(x)V(y)\rangle = W\delta(x - y)$. We will mostly be interested in two dimensions ($D = 2$), but will leave a general $D$ in the action to comment later on results in other dimensions. Note that disorder is assumed to be a random function only of spatial coordinates, while it is completely correlated in (i.e. independent of) imaginary time. This is what makes it much stronger than in the corresponding problem in classical mechanics. The theory (1) for $N = 2$ describes the superfluid order parameter in the Bose–Hubbard model, at a density of bosons commensurate with the lattice [2][1], also known in the literature as the random-rod problem [8]. For $N = 3$ the theory may be used to describe disordered quantum rotors, i.e. the magnetic quantum phase transitions in the Heisenberg universality class in the presence of quenched randomness [9]. When $N = 1$ the theory describes a random system with the Ising symmetry. In general the action (1) provides a minimal description of the quantum disordered interacting system, and for $N = \infty$ has been studied by renormalization group methods in the past [10, 11] with conflicting results. The purpose of this paper is to shed some light on the physics implicit in this model and, in particular, to argue that the model allows no superfluid phase in $D = 2$.

To see what is involved in solving the problem in the (spherical) limit $N = \infty$, perform the standard Hubbard–Stratonovich transformation on the quartic term and integrate all but one of the bosonic fields. This leaves one with the transformed action

$$S_{\text{eff}}[\chi, \psi] = \int d^D\boldsymbol{x}\, d\tau \left\{ -\frac{N}{4\lambda}\chi^2(\boldsymbol{x}, \tau) + (\partial_\tau \Psi_1(\boldsymbol{x}, \tau))^2 \right.$$
$$+ (\nabla \Psi_1(\boldsymbol{x}, \tau))^2 + (V(\boldsymbol{x}) + \chi(\boldsymbol{x}, \tau) - \mu)\, \Psi_1^2(\boldsymbol{x}, \tau) \bigg\}$$
$$+ \frac{1}{2}(N - 1)\ln\det\left\{ -\partial_\tau^2 - \nabla^2 + V(\boldsymbol{x}) + \chi(\boldsymbol{x}, \tau) - \mu \right\} \tag{2}$$

which is just the original problem rewritten exactly. Assuming that the Hubbard–Stratonovich field at the saddle point is independent of imaginary time, $\chi(\boldsymbol{x}, \tau) = \chi(\boldsymbol{x})$, and that $\Psi_1(\boldsymbol{x}, \tau) = N^{1/2}c\phi_0(\boldsymbol{x})$, the saddle-point equations become

$$\chi(\boldsymbol{x}) = \lambda \left\langle \boldsymbol{x}, \tau \left| \frac{1}{-\partial_\tau^2 - \nabla^2 + V(\boldsymbol{x}) + \chi(\boldsymbol{x}) - \mu} \right| \boldsymbol{x}, \tau \right\rangle + c^2\phi_0^2(\boldsymbol{x}) \tag{3}$$

$$\varepsilon_0 c = 0 \tag{4}$$

where $\phi_\alpha(\boldsymbol{x})$ are the random eigenstates and $\varepsilon_\alpha$ the random eigenvalues of the susceptibility matrix

$$(-\nabla^2 + V(\boldsymbol{x}) + \chi(\boldsymbol{x}) - \mu)\phi_\alpha(\boldsymbol{x}) = \varepsilon_\alpha\phi_\alpha(\boldsymbol{x}) \tag{5}$$

with $\varepsilon_0$ being the lowest eigenvalue. Equations (3)–(5) are completely standard and the only novelty compared to the case without disorder [12] is the random spectrum instead of the usual plane waves. In principle, one may expect to have two phases: $\varepsilon_0 \neq 0$ and $c = 0$, corresponding to the gapped Mott insulator (MI), or $\varepsilon_0 = 0$ and $c \neq 0$ which would represent a superfluid (SF). The gapless insulating Bose-glass (BG) between the MI and the SF [2] should in general be absent, as we argue below.

With $V(\boldsymbol{x}) = 0$, the solution $\chi(\boldsymbol{x}) = \chi_0$ is uniform and the model leads to the well-known large-N critical behavior in $D + 1$ dimensions [12]. The correlation length exponent, for

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[1] More precisely, for commensurate dirty bosons the action contains an additional term linear in the time derivative with a random coefficient with zero average. We neglect this additional disorder for simplicity.

example, in the pure case is $\nu = 1/(D-1)$, and in $D = 2$ the Harris criterion [13] (which says that disorder is irrelevant if $\nu D > 2$) implies that disorder is precisely *marginal*. When $V(x) \neq 0$, in the MI phase the saddle point equation (3), after integration over the frequency, can be written in the basis $\{\phi_\alpha\}$ as

$$\chi(x) = \lambda \sum_\alpha \frac{\phi_\alpha(x)^2}{\sqrt{\varepsilon_\alpha}}. \tag{6}$$

The functions $\phi_\alpha$ are the eigenstates of the *screened*, but nevertheless random, potential $V(x) + \chi(x)$ and all would therefore naively be expected to be localized in $D = 2$ [7]. In particular, for the localized ground state the first term in the sum in equation (6) becomes large as $\varepsilon_0 \to 0$ precisely in the region of localization, which by self-consistency implies that $\chi(x)$ is also large there. This, on the other hand, then implies that $\varepsilon_0$ is large, and not small as assumed and one runs into a contradiction. Evidently, for the spectrum to extend all the way to zero, the discrete sum in the last equation must be approximable by an integral so that the infrared singularity becomes integrable. For this to occur the weight of each of the terms corresponding to the low-energy states in equation (6) must vanish in the thermodynamic limit as the inverse of the system size, which is tantamount to delocalization of the low-energy eigenstates. Put differently, the collapse of the gap must be accompanied by the simultaneous delocalization of the ground state, so that the gapless phase is necessarily a SF. There can be no intermediate localized BG in the model at $N = \infty$.

   With this picture in mind the appearance of the superfluid phase in the large-N model in $D = 2$ appears rather counter-intuitive: although screening introduces correlations into the effective random potential, the states should nevertheless always remain localized. In the rest of the paper we first show that although the lowest order screening does reduce the random potential, it does not make it completely smooth and consequently the MI gap cannot close. This conclusion is further corroborated by the numerical solution of the self-consistent equations on a lattice and absence of the finite-size scaling of the gap and the ground state participation ratio. In the closing section we compare our result with other studies and speculate on the implications for physical cases $N = 1, 2, 3$.

## 2. Weak-disorder expansion

For a given random configuration the self-consistent equations cannot be solved analytically and one has to resort to numerical computations. For weak disorder, however, we can expand the matrix element in (3) in powers of the *screened* potential. To that end write $\chi(x) = \chi_0 + \chi_1(x)$, where $\int \chi_1(x) d^D x = 0$. The uniform part $\chi_0$ is just the renormalization of the chemical potential, while $\tilde{V}(x) \equiv V(x) + \chi_1(x)$ is the screened potential, which should vanish with vanishing randomness. Expanding the right hand side of (3) in the MI phase in $\tilde{V}(x)$ and taking the Fourier transform, we get (for $q \neq 0$)

$$\tilde{V}(q) = V(q) - \lambda \Pi(q) \tilde{V}(q) + \lambda \int dk\, I_1(k, q) \tilde{V}(k) \tilde{V}(q - k)$$

$$- \lambda \int dk\, dl\, I_2(k, l, q) \tilde{V}(k) \tilde{V}(l) \tilde{V}(q - k - l) + \mathcal{O}(\tilde{V}^4) \tag{7}$$

where

$$\Pi(q) \equiv \int dp\, d\omega\, G_0(\omega, p) G_0(\omega, p + q) \tag{8}$$

is the standard polarization bubble, and

$$I_1(k, q) \equiv \int dp\, d\omega\, G_0(\omega, p) G_0(\omega, p + k) G_0(\omega, p + q) \tag{9}$$
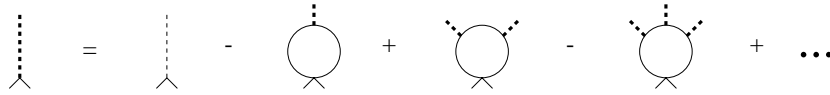
**Figure 1.** Diagrammatic representation of equation (7). The heavy dashed line represents the self-consistently screened random potential, while the thin dashed line is the bare random potential.
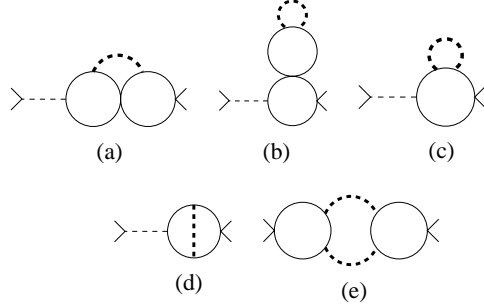


**Figure 2.** Diagrams corresponding to the second-order terms in expansion (11).

and

$$I_2(\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{q}) \equiv \int \mathrm{d}\boldsymbol{p}\, \mathrm{d}\omega\, G_0(\omega, \boldsymbol{p}) G_0(\omega, \boldsymbol{p}+\boldsymbol{k}) G_0(\omega, \boldsymbol{p}+\boldsymbol{k}+\boldsymbol{l}) G_0(\omega, \boldsymbol{p}+\boldsymbol{q}). \tag{10}$$

The propagator for the clean case is given by $G_0^{-1}(\omega, \boldsymbol{p}) = \omega^2 + p^2 + \Omega^2$, where $\Omega^2 \equiv \chi_0 - \mu > 0$ and is the MI gap. Equation (7) can be represented diagrammatically as in figure 1.

We next introduce the two-point correlator $\widetilde{W}(\boldsymbol{q})\delta(\boldsymbol{r}) = \langle \tilde{V}(\boldsymbol{q})\tilde{V}(-\boldsymbol{q}+\boldsymbol{r})\rangle$, where $\langle \cdots \rangle$ represents disorder averaging as a measure of the screened disorder. From equation (7) it follows that

$$\widetilde{W}(\boldsymbol{q})\{1 + \lambda \Pi(\boldsymbol{q})\}^2 = W(\boldsymbol{q}) + 2\lambda \int \mathrm{d}\boldsymbol{k}\, I_1(\boldsymbol{k}, \boldsymbol{q})\langle V(-\boldsymbol{q})\tilde{V}(\boldsymbol{k})\tilde{V}(\boldsymbol{q}-\boldsymbol{k})\rangle$$

$$- 2\lambda \int \mathrm{d}\boldsymbol{k}\, \mathrm{d}\boldsymbol{l}\, I_2(\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{q})\langle V(-\boldsymbol{q})\tilde{V}(\boldsymbol{k})\tilde{V}(\boldsymbol{l})\tilde{V}(\boldsymbol{q}-\boldsymbol{k}-\boldsymbol{l})\rangle$$

$$+ \lambda^2 \int \mathrm{d}\boldsymbol{k}\, \mathrm{d}\boldsymbol{k}'\, I_1(\boldsymbol{k}, \boldsymbol{q})I_1(\boldsymbol{k}', -\boldsymbol{q})\langle \tilde{V}(\boldsymbol{k})\tilde{V}(\boldsymbol{q}-\boldsymbol{k})\tilde{V}(\boldsymbol{k}')\tilde{V}(-\boldsymbol{q}-\boldsymbol{k}')\rangle$$

$$+ \mathcal{O}(W^3). \tag{11}$$

Diagrammatically, the second-order contributions may be represented as in figure 2. In the appendix we compute the above averages in $D = 2$. Note that although the random potential is assumed to be uncorrelated in space, the screened potential develops correlations and $\widetilde{W}(\boldsymbol{q})$ becomes a non-trivial function of the wave vector. For low-energy states one expects the localization properties to be determined by $\widetilde{W}(\boldsymbol{q})$ at small $\boldsymbol{q}$, so we focus on the limit $\boldsymbol{q} \to 0$ and denote $\widetilde{W}(\boldsymbol{q} \to 0) = \widetilde{W}$. To the second order in $W$ in the limit $\Omega \to 0$ and in $D = 2$ one then finds (see the Appendix for details):

$$\widetilde{W} = \frac{W}{\lambda^2 c^2}\Omega^2 + \left(\frac{W}{\lambda^2 c^2}\right)^2 \Omega^2$$

$$\times \left\{ \frac{1}{2\pi^5}\left(\left(\frac{\Lambda}{\Omega}\right)^2 + \frac{32}{\pi}\left(\frac{\Lambda}{\Omega}\right)\right) + \frac{4}{\pi^4}\left(\frac{\Lambda}{\Omega}\right) + \mathcal{O}\left(\ln\left(\frac{\Lambda}{\Omega}\right)\right) \right\} + O(W^3) \tag{12}$$

where the constant $c = 1/(8\pi)$ and $\Lambda$ is the ultraviolet cut-off implicit in (7).

The last equation is our central result and several remarks are in order. To the first order in $W$, one finds that as $\Omega \to 0$, $\widetilde{W} \to 0$, which one may be tempted to interpret as a sign of delocalization of the ground state. This is a consequence of the screening of the random potential by the medium, which to the zeroth order in disorder is pure and thus screens perfectly at $q = 0$. Also, recognizing the combination $\widetilde{W}/\Omega^2$ as a dimensionless measure of screened disorder, to the lowest order equation (12) agrees with the Harris criterion: disorder is marginal in $D = 2$. The fate of disorder is therefore determined by the higher order terms in the expansion. To the second order in disorder we find that

$$\widetilde{W} \to \frac{\Lambda^2}{2\pi^5}\left(\frac{W}{\lambda^2 c^2}\right)^2, \qquad \text{as } \Omega \to 0 \tag{13}$$

i.e. goes to a non-universal finite constant as the gap decreases. If the bare disorder is weak the screened disorder will be even weaker, but always finite. The consequence is that the ground state and the excited states in $D = 2$ should remain localized [14] so that our qualitative argument from the introduction would imply that the gap cannot close. This is in accordance with the direct numerical solution at strong disorder to which we turn next.

## 3. Numerical solution

We begin by introducing a discrete version of our theory where the continuous variable $x$ is replaced by a lattice-site index $i$ on a quadratic lattice of linear size $L$. The kinetic energy term $\nabla^2$ is replaced by the nearest-neighbour hopping measured by $t$, the random potential is chosen from a uniform distribution of width $W$ and the interaction strength is given by $\lambda$. In our calculations we set $W/t = 4$ and $\lambda/t = 8$, which correspond to strong disorder and interactions. After the integration over frequency, the self-consistent equation (3) becomes

$$\chi_i = \lambda \sum_{\alpha=1}^{N} \frac{\phi_\alpha(i)^2}{\sqrt{\varepsilon_\alpha}} \tag{14}$$

on the $\{\phi_\alpha(i)\}$ basis where these wave functions are now eigenvectors of the matrix

$$\sum_j \{-t_{ij} + (V_i + \chi_i - \mu)\,\delta_{i,j}\}\phi_\alpha(j) = \varepsilon_\alpha \phi_\alpha(i) \tag{15}$$

where $t_{ij}$ is non-zero for nearest-neighbour $i, j$ only.

We solve the set of $L \times L$ equations using the Newton–Raphson algorithm. We gradually increase the chemical potential $\mu$, using the last found solution as the initial guess at the next $\mu$. Finally, we average over many disorder configurations. Of course, for finite $L$ the gap is always finite, so to infer the result in the thermodynamic limit we make the standard finite-size scaling ansatz for the average ground state energy

$$\varepsilon_0 = L^{-z} F\left[L^{\frac{1}{\nu}}(\mu - \mu_c)\right] \tag{16}$$

where $z$ is the *dynamical* critical exponent, $\nu$ is the correlation length exponent and $\mu_c$ is the critical point in the thermodynamic limit; $F(x)$ is a universal scaling function. The values of $z$ and $\mu_c$ are determined by scaling the $\varepsilon_0$-axis until all curves cross at a single point. The exponent $\nu$ is found by scaling the $\mu$-axis so that a reasonable collapse of all the data onto a single curve is achieved.

Such an attempt of finite size scaling of our data is shown in figure 3 for systems of linear size $L = 6, 8, 10, 12$. We display the result for the value $z = 0.9$, but the picture remains qualitatively the same for all $0.5 < z < 1.0$. The gap continuously decreases
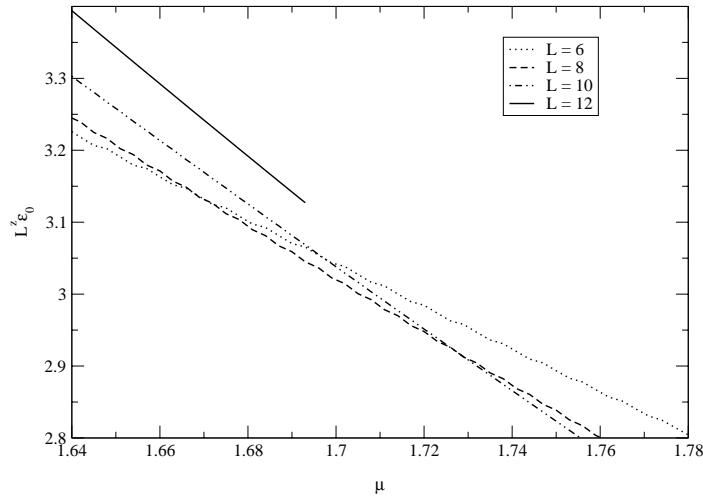
**Figure 3.** Finite-size scaling attempt of the ground state energy $\varepsilon_0$ with $z = 0.9$ demonstrating the lack of transition in our numerical calculations. The disorder averaging was done over 500 configurations for $L = 6$, 1200 for $L = 8$, for $L = 10$ and 1000 for $L = 10$.

with $\mu$, but the failure of finite-size scaling suggests that it does not vanish in the thermodynamic limit.

We have also argued that at the point of collapse of the gap, the ground state would be expected to become delocalized. A useful measure of the degree of localization of the wavefunctions at a given energy is provided by the participation ratio

$$P(\varepsilon) = \sum_\alpha \frac{\delta(\varepsilon_\alpha - \varepsilon)}{L^2 \sum_{i=1}^{L^2} |\phi_\alpha(i)|^4} \tag{17}$$

which is proportional to $1/L^2$ for the localized states and approaches a constant for the extended states. In the critical region, one expects the participation ratio to assume a similar finite-size scaling form

$$P(\varepsilon_0) = L^{-(D-D_f)} \Phi \left[ L^{\frac{1}{\nu}}(\mu - \mu_c) \right] \tag{18}$$

where $D_f$ is the fractal dimension of the ground state wavefunction and $\Phi(x)$ another scaling function. Our data for the participation ratio are shown in figure 4 for the sizes $L = 8, 10, 12$. Again, attempts to find a common crossing point by tuning $D_f$ fail. We see that the participation ratio of the ground state grows as $\mu$ increases, but conclude that the ground state nevertheless seems to remain localized. This is consistent with the data for the ground state energy.

Our Newton–Raphson algorithm has difficulties converging as $\mu$ increases and the problem becomes more non-linear. It is possible we simply have not been able to reach the critical point in our numerical calculation. When taken together with the weak-disorder expansion and the physical arguments, however, we believe a more likely interpretation is that there is no SF phase in the model.
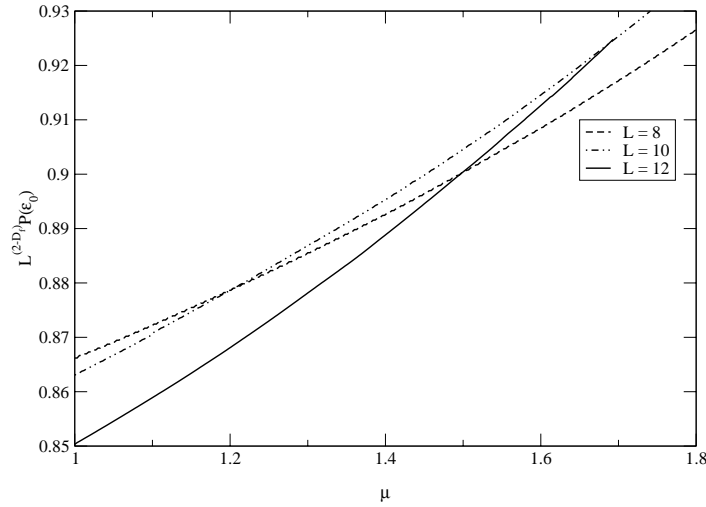
**Figure 4.** Finite-size scaling attempt of the ground state participation ratio with $D_f = 0.5$. Again, the inability to cross these curves at a common point indicates the lack of the transition. Disorder averaging was done over the same configurations as in figure A3.

## 4. Conclusion

To summarize, we studied the large-*N* limit of the commensurate dirty boson theory and argued that at weak disorder screening does not delocalize the ground state and consequently, that there is no MI–SF transition in $D = 2$. Numerical results for the ground state energy and the participation ratio that support this conclusion were provided.

Our conclusion agrees with the results of Kim and Wen [10] who found that disorder is always relevant for $D \geqslant 2$ and could not find any stable critical points within their renormalization scheme. The latter point may, in principle, be interpreted in three ways: as a failure of the renormalization procedure, as the transition that is discontinuous, or that there is no transition. Our findings support the third conclusion. On the other hand, we are in disagreement with the recent study of Hastings [11], who considered the closely related random spherical model and found the disorder to be marginally *irrelevant* in $D = 2$. At the moment we do not fully understand what is the resolution of this disagreement, nor how the ground state becomes extended in Hastings' theory.

While we were mostly concerned with $D = 2$, the same perturbative procedure can be repeated in $D = 3$. We found that the same diagram in figure 2(*e*) that led to the finite term for $\widetilde{W}$ in $D = 3$ vanishes logarithmically as the gap decreases. More importantly, in $D = 3$, the Anderson localization problem allows a mobility edge, so the screened disorder need not go all the way to zero for the ground state to delocalize. We would therefore expect that theory (1) would have a MI–SF transition in $D = 3$, as apparently has been found in earlier numerical calculations [15].

An important question is what our considerations imply for the physical cases $N = 1, 2, 3$ mentioned in the introduction. We believe that in $D = 2$, for $N = 2$ theory (1) does have a transition and which is in the BG–SF universality class. This has been found in the dual theory for the commensurate dirty bosons [3], in both $D = 1$ and $D = 2$ and in detailed numerical calculations [5, 16]. The BG–SF transition is best understood in terms of disorder-induced proliferation of topological defects and thus is very specific to having a complex ($N = 2$)

order parameter. The same topological mechanism will not apply to the case of a random quantum ferromagnet $N = 3$ and we conjecture that for $N = 3$ there may not be a gapless phase in $D = 2$. On the same grounds, we expect that for the Ising case $N = 1$ the transition again will exist [17].

Finally, we note the similarity between our problem and that of interacting disordered fermions in $D = 2$ [18]. In the large-N limit the metallic phase in the fermionic problem would correspond to an extended state at the Fermi level, as opposed to the extended ground state in our problem. Nevertheless, one can show [19] that already to the lowest order in disorder, screened disorder remains finite and thus the state should remain localized. We would therefore expect that the fermionic version of action (1) also should have only the localized phase in $D = 2$, at least in the large-N limit.

## Acknowledgment

## Appendix A.

In this appendix, we provide the details of the calculations leading up to our main analytic result (12). We begin by calculating the integrals (8)–(10). Using the standard Feynman parameters [12] the integrals can be rewritten as

$$
\Pi(q) = \frac{\Gamma\left(\frac{3-D}{2}\right)}{(4\pi)^{\frac{D+1}{2}}} \int_0^1 dt \frac{1}{\left[t(1-t)q^2 + \Omega^2\right]^{\frac{3-D}{2}}}
$$

$$
\rightarrow \begin{cases} \frac{c}{\Omega}\left[\pi\left(\frac{\Omega}{q}\right) - 4\left(\frac{\Omega}{q}\right)^2 + \mathcal{O}\left(\left(\frac{\Omega}{q}\right)^3\right)\right] & \frac{q}{\Omega} \rightarrow \infty \\ \frac{c}{\Omega} & q \rightarrow 0 \end{cases} \tag{A.1}
$$

$$
I_1(k, 0) = \frac{1}{2}\frac{\Gamma\left(\frac{5-D}{2}\right)}{(4\pi)^{\frac{D+1}{2}}} \int_0^1 dt \frac{1}{\left[t(1-t)k^2 + \Omega^2\right]^{\frac{5-D}{2}}}
$$

$$
\rightarrow \begin{cases} \frac{c}{4\Omega^3}\left[4\left(\frac{\Omega}{k}\right)^2 - 16\left(\frac{\Omega}{k}\right)^4 + \mathcal{O}\left(\left(\frac{\Omega}{k}\right)^6\right)\right] & \frac{k}{\Omega} \rightarrow \infty \\ \frac{c}{4\Omega^3} & k \rightarrow 0. \end{cases} \tag{A.2}
$$

where we assumed $D = 2$ in evaluating the limits. The diagrams in figures 2(c, d) will require the evaluation of the following two limits of $I_2$:

$$
I_2(k, 0, 0) = \frac{\Gamma\left(\frac{7-D}{2}\right)}{(4\pi)^{\frac{D+1}{2}}} \int_0^1 dt \frac{t(1-t)}{\left[t(1-t)k^2 + \Omega^2\right]^{\frac{7-D}{2}}}
$$

$$
\rightarrow \frac{3c}{4\Omega^5}\left[\frac{8}{3}\left(\frac{\Omega}{k}\right)^4 + \mathcal{O}\left(\left(\frac{\Omega}{k}\right)^6\right)\right] \qquad \frac{k}{\Omega} \rightarrow \infty \tag{A.3}
$$

$$I_2(0, \boldsymbol{l}, 0) = \frac{1}{2} \frac{\Gamma\left(\frac{7-D}{2}\right)}{(4\pi)^{\frac{D+1}{2}}} \int_0^1 \mathrm{d}t \frac{(1-t)^2}{\left[t(1-t)l^2 + \Omega^2\right]^{\frac{7-D}{2}}}$$

$$\rightarrow \frac{3c}{8\Omega^5} \left[\frac{2}{3}\left(\frac{\Omega}{l}\right)^2 + \mathcal{O}\left(\left(\frac{\Omega}{l}\right)^6\right)\right] \qquad \frac{l}{\Omega} \rightarrow \infty \qquad (A.4)$$

where the limits $\frac{P}{\Omega} \rightarrow \infty$ and $p \rightarrow 0$ are taken with fixed $p$ and $\Omega$, respectively, and in $D = 2$. We also define $c \equiv \Gamma(\frac{1}{2})/(4\pi)^{3/2} = 1/(8\pi)$.

We can now evaluate the series (11) term by term. From the first-order term, in the limit $\boldsymbol{q} \rightarrow 0$, we get

$$\widetilde{W}_1(\boldsymbol{q} \rightarrow 0) \equiv \frac{W}{(1 + \lambda\Pi(0))^2} = \frac{W}{\lambda^2 c^2}\Omega^2. \qquad (A.5)$$

The contributions of order $\mathcal{O}(W^2)$ are shown diagrammatically in figure 2. Referring to this figure, we label the corresponding terms generated in the expansion accordingly. To illustrate our procedure, we will explicitly calculate the diagram shown in figure 2(*e*) arising from the final term in (11). This term is

$$\widetilde{W}_{2(e)}(\boldsymbol{q}) = \frac{\lambda^2}{\{1 + \lambda\Pi(\boldsymbol{q})\}^2} \int \mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{k}'\, I_1(\boldsymbol{k}, \boldsymbol{q}) I_1(\boldsymbol{k}', -\boldsymbol{q})$$

$$\times \langle \tilde{V}(\boldsymbol{k})\tilde{V}(\boldsymbol{q} - \boldsymbol{k})\tilde{V}(\boldsymbol{k}')\tilde{V}(-\boldsymbol{q} - \boldsymbol{k}')\rangle \qquad (A.6)$$

where

$$\langle \tilde{V}(\boldsymbol{k})\tilde{V}(\boldsymbol{q} - \boldsymbol{k})\tilde{V}(\boldsymbol{k}')\tilde{V}(-\boldsymbol{q} - \boldsymbol{k}')\rangle = 2\langle \tilde{V}(\boldsymbol{k})\tilde{V}(\boldsymbol{k}')\rangle\langle \tilde{V}(\boldsymbol{q} - \boldsymbol{k})\tilde{V}(-\boldsymbol{q} - \boldsymbol{k}')\rangle \qquad (A.7)$$

are the contractions which contribute to $\boldsymbol{q} \neq 0$. Using the definition

$$\widetilde{W}(\boldsymbol{q})\delta(r) \equiv \langle \tilde{V}(\boldsymbol{q})\tilde{V}(-\boldsymbol{q} + \boldsymbol{r})\rangle \qquad (A.8)$$

and integrating over $\boldsymbol{k}'$, (A.6) becomes

$$\widetilde{W}_{2(e)}(\boldsymbol{q}) = \frac{2\lambda^2}{\{1 + \lambda\Pi(\boldsymbol{q})\}^2} \int \mathrm{d}\boldsymbol{k}\, I_1(\boldsymbol{k}, \boldsymbol{q}) I_1(-\boldsymbol{k}, -\boldsymbol{q}) \widetilde{W}(\boldsymbol{k}) \widetilde{W}(\boldsymbol{q} - \boldsymbol{k})$$

$$\rightarrow \frac{2\lambda^2}{\{1 + \lambda\Pi(0)\}^2} \int \mathrm{d}\boldsymbol{k}\, I_1^2(\boldsymbol{k}, 0) \widetilde{W}^2(\boldsymbol{k}) \qquad \boldsymbol{q} \rightarrow 0. \qquad (A.9)$$

We now replace $\widetilde{W}(\boldsymbol{k})$ in (A.9) to first order in $W$ to get

$$\widetilde{W}_{2(e)}(\boldsymbol{q} \rightarrow 0) = \frac{2\lambda^2 W^2}{\{1 + \lambda\Pi(0)\}^2} \int \mathrm{d}\boldsymbol{k}\, \frac{I_1^2(\boldsymbol{k}, 0)}{\{1 + \lambda\Pi(\boldsymbol{k})\}^4}$$

$$= \frac{1}{\pi^5}\left(\frac{W}{\lambda^2 c^2}\right)^2 \Omega^2 \int_0^{\Lambda/\Omega} x\,\mathrm{d}x\, \frac{\left[1 - 4\left(\frac{1}{x^2}\right) + \mathcal{O}\left(\frac{1}{x^4}\right)\right]}{\left[1 - \frac{4}{\pi}\left(\frac{1}{x}\right) + \mathcal{O}\left(\frac{1}{x^2}\right)\right]} \qquad (A.10)$$

where the last line follows from substituting (A.1) and (A.2) into the previous line and making the change of variable $x = k/\Omega$; $\Lambda$ is the usual ultraviolet cut-off imposed by the lattice. Expanding the denominator in (A.10) and integrating over $x$ now yields the result

$$\widetilde{W}_{2(e)}(\boldsymbol{q} \rightarrow 0) = \frac{1}{2\pi^5}\left(\frac{W}{\lambda^2 c^2}\right)\Omega^2\left[\left(\frac{\Lambda}{\Omega}\right)^2 + \frac{32}{\pi}\left(\frac{\Lambda}{\Omega}\right) + \mathcal{O}\left(\ln\left(\frac{\Lambda}{\Omega}\right)\right)\right]. \qquad (A.11)$$

It is important to note that (A.11) does not vanish as $\Omega \rightarrow 0$.

The calculation of the remaining terms now follows in a similar way. The diagrams arising from the second term on the RHS of (11) are those shown in figures 2(a, b). These give

$$\widetilde{W}_{2(a)}(\boldsymbol{q} \to 0) = \frac{4}{\pi^4} \left( \frac{W}{\lambda^2 c^2} \right)^2 \Omega^2 \left[ \left( \frac{\Lambda}{\Omega} \right) + \mathcal{O} \left( \ln \left( \frac{\Omega}{\Lambda} \right) \right) \right] \qquad (A.12)$$

and

$$\widetilde{W}_{2(b)}(\boldsymbol{q} \to 0) = \frac{1}{4\pi^3} \left( \frac{W}{\lambda^2 c^2} \right)^2 \Omega^2 \left[ \left( \frac{\Lambda}{\Omega} \right)^2 + \frac{16}{\pi} \left( \frac{\Lambda}{\Omega} \right) + \mathcal{O} \left( \ln \left( \frac{\Lambda}{\Omega} \right) \right) \right]. \qquad (A.13)$$

The third term on the RHS of (11) gives rise to the diagrams 2(c, d). These give

$$\widetilde{W}_{2(c)}(\boldsymbol{q} \to 0) = -\frac{1}{4\pi^3} \left( \frac{W}{\lambda^2 c^2} \right)^2 \Omega^2 \left[ \left( \frac{\Lambda}{\Omega} \right)^2 + \frac{16}{\pi} \left( \frac{\Lambda}{\Omega} \right) + \mathcal{O} \left( \ln \left( \frac{\Lambda}{\Omega} \right) \right) \right] \qquad (A.14)$$

and

$$\widetilde{W}_{2(d)}(\boldsymbol{q} \to 0) = -\frac{2}{\pi^3} \left( \frac{W}{\lambda^2 c^2} \right)^2 \Omega^2 \left[ \mathcal{O} \left( \ln \left( \frac{\Lambda}{\Omega} \right) \right) \right]. \qquad (A.15)$$

Note that the two highest order terms in (A.14) cancel exactly with those in (A.13). Summing the contributions (A.5) and (A.11)–(A.15), we then get the result quoted in (12). As mentioned, the second-order term that remains constant when $\Omega \to 0$ comes entirely from the diagram 2(e).

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